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Romain Guillaume, Caroline Thierry, Pawel Zielinski

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# Robust Production Plan with Periodic Order Quantity under Uncertain Cumulative Demands

Romain Guillaume
Université de Toulouse-IRIT
5, Allées A. Machado
31058 Toulouse Cedex 1, France
Email: Romain.Guillaume@irit.fr

Caroline Thierry
Université de Toulouse-IRIT
5, Allées A. Machado
31058 Toulouse Cedex, France
Email: Caroline.Thierry@irit.fr

Paweł Zieliński
Institute of Mathematics
and Computer Science
Wrocław University of Technology
Wybrzeże Wyspiańskiego 27
50-370 Wrocław, Poland
Email: Pawel.Zielinski@pwr.wroc.pl

***Abstract*—In this paper, we are interested in a production planning process in collaborative supply chains. More precisely, we consider supply chains, where actors use Manufacturing Resource Planning process (MRPII). Moreover, these actors collaborate by sharing procurement plans. We focus on a supplier, who applies the Periodic Order Quantity (POQ) rule to plan a production integrating the uncertain procurement plan sent by her/his customer. The uncertainty of the procurement plan is expressed by closed intervals on the cumulative demands. In order to choose a robust production plan, under the interval uncertainty representation, the min-max criterion is applied.**

**We propose algorithms for determining the set of possible costs of a given production plan - due to the uncertainty on the cumulative demands. We then construct algorithms for computing a robust production plan with respect to the min-max criterion: the algorithm based on iterative adding constraints and the polynomial algorithms under certain realistic assumptions.**

***Index Terms*—Supply Chain, Production Planning, Uncertainty, Scenario Optimization.**

## I. Introduction

Companies today evolve in high competitive context that obliges the companies to collaborate with their suppliers and customers and creates uncertainty on the demand. Due to the well-known bullwhip effect [1], this uncertainty induces supply chain risks as backordering, obsolete inventory. Sharing information on the demand and the collaboration with the suppliers are ways to reduce this risk.

Most companies use Manufacturing Resource Planning (MRPII) to plan and control all resources of a manufacturing company. MRPII is composed of three processes (the production process, the procurement process and the distribution process) and three levels [2]: the strategic level (Sales and Operation Plan-S&OP), the tactical level (Master Production Scheduling (MPS) and Material Requirement Planning (MRP)) and the operational level (detailed scheduling and shop floor control). Within MRP process, different lot sizing rules exist for purchased or produced items, as Fixed Order Quantity (FOQ), Lot-for-Lot (L4L), Minimal Order Quantity (MOQ), Periodic Order Quantity (POQ), etc. In collaborative supply chains, collaboration is usually characterized by a set of point-to-point (customer/supplier) relationships with partial information sharing. More precisely, the collaboration process in supply chains, where actors use MRPII, is realized by sharing procurement plans through the supply chain. The procurement plan can take into account uncertainty [3]. Thus, the problem is how to integrate this information in a production planning process. In this paper, we focus on cases where no probability distribution is available to model the uncertainty. In this context, the uncertainty is modeled by specifying a set of all possible realizations of the demand, called scenarios.

In the literature, the planning processes of MRPII have been extended to take into account the imprecision on quantities of period demands (MPS and MRP) [4], [5], [6], [7], quantities of period demands and uncertain orders (MRP) [8] and the imprecision on order quantities and dates with uncertain order (MRP) [9].

To deal with the uncertainty in the production planning, three approaches can be distinguished: computing the possible inventory and backordering levels over all scenarios to help the decision maker to choose a production plan ([4], [8], [9]), computing an optimal solution for one of possible demand scenarios [5], [6] and the robust optimization [10] under the scenario uncertainty representation, more precisely using the min-max criterion [7]. Under this criterion, we seek a solution that minimizes the largest cost over all scenarios. The cost function in production planning is the sum of inventory and backordering costs over the planning horizon.

The aim of this paper is to investigate the MRP process with the POQ rule under imprecision on cumulative demands. The POQ rule consists in producing a quantity equal to the gross requirements for $P$ periods minus any items in on-hand inventory plus any additional items needed to replenish safety stock if it has fallen below its desired level. To adapt the MRP with the POQ rule to the uncertain context, we have to consider the problem with backordering. Indeed, the problem without backordering is not satisfactory due to the fact that a solution method (without backordering) consists in applying the rule to the maximal cumulative demands. Thus, this solution method induces too much inventory and does not consider preferences of the decision maker between possible inventory and backordering levels. In the model proposed in this paper, the imprecision on the demand is represented by cumulative demand intervals. Such modeling allows us to describe the imprecision on order quantities and dates.

The paper is organized as follows. Section II presents the problem under consideration with the precise demands (parameters). In Section III, we formulate the problem under the scenario uncertainty model in the robust optimization setting. We adopt min-max criterion to choose a robust production plan. We then propose algorithms for evaluating a given production plan (for determining optimal interval containing all possible values of costs of the production plan) and for computing an optimal robust production plan.

## II. The Deterministic Problem

In this section, a deterministic version of the problem under consideration, i.e. the problem in which all parameters are precisely known in advance.

Given $T+1$ periods. For period $t$, $t=0,\dots,T$, let $d_t$ be the demand in period $t$, $d_t \geq 0$, $x_t$ the production amount in period $t$. Furthermore, we are given a periodicity $P$, $P \in \mathbb{N}$, such that: $x_t \geq 0$ if $t = k \cdot P$; otherwise (if $t \neq k \cdot P$) $x_t = 0$ for $k = 0,\dots,N$ and $t = 0,\dots T$, where $N = T/P$ (we assume without loss of generality that $T$ is divisible by $P$). Now, the set feasible production amounts $\mathbb{X} \subseteq \mathbb{R}^{T+1}_{\geq 0}$ can be defined as follows:

$$\begin{aligned}\mathbb{X} = \{\boldsymbol{x} = (x_0,\dots,x_T) \ : \ & x_t \geq 0 \text{ for } t = k \cdot P,\\ & x_t = 0 \text{ for } t \neq k \cdot P,\\ & k = 0,\dots,N, t = 0,\dots,T\}.\end{aligned}$$

Set $\mathbf{D}_t = \sum_{i=0}^{t} d_i$ and $\mathbf{X}_t = \sum_{i=0}^{t} x_i$, $\mathbf{D}_t$ and $\mathbf{X}_t$ stand for the cumulative demand up to period $t$ and the production level up to period $t$, respectively. Obviously, $\mathbf{X}_{t-1} \leq \mathbf{X}_t$ and $\mathbf{D}_{t-1} \leq \mathbf{D}_t$, $t = 1,\dots,T$. The nonnegative costs of carrying one unit of inventory from period $t$ to period $t+1$ are given, denoted by $c^I$, and all the inventory costs are equal for every period. The nonnegative costs of backordering one unit from period $t+1$ to period $t$ are given, denoted by $c^B$, and all the backorder costs are equal for every period. Furthermore, we assume that $c^I \leq c^B$. The nonnegative real function $C_t(u,v)$ represents either the cost of storing inventory from period $t$ to period $t+1$ or the cost of backordering quantity from period $t+1$ to period $t$, namely $C_t(\mathbf{X}_t, \mathbf{D}_t) = c^I(\mathbf{X}_t - \mathbf{D}_t)$ if $\mathbf{X}_t \geq \mathbf{D}_t$; $c^B(\mathbf{D}_t - \mathbf{X}_t)$ otherwise. The function has the form $C_t(\mathbf{X}_t, \mathbf{D}_t) = \max\{c^I(\mathbf{X}_t - \mathbf{D}_t), c^B(\mathbf{D}_t - \mathbf{X}_t)\}$.

The optimization problem with the precise parameters consists in finding a feasible production plan $\boldsymbol{x} = (x_0,\dots,x_T)$, $\boldsymbol{x} \in \mathbb{X}$, that minimizes the total cost of storage and backordering subject to the conditions of satisfying each demand, that is

$$\begin{aligned}\min_{\boldsymbol{x}\in\mathbb{X}} F(\boldsymbol{x}) &= \min_{\boldsymbol{x}\in\mathbb{X}} \sum_{t=0}^{T} C_t\left(\sum_{i=0}^{t} x_i, \sum_{i=0}^{t} d_i\right)\\ &= \min_{\boldsymbol{x}\in\mathbb{X}} \sum_{t=0}^{T} C_t(\mathbf{X}_t, \mathbf{D}_t). \qquad (1)\end{aligned}$$

It is easily seen that when $P = 1$, the problem (1) is equivalent to the *classical lot sizing with backordering* with the Lot-For-Lot (L4L) rule (see, e.g., [11], [12], [13]). The problem (1) can be formulated as the minimum cost flow problem (see, e.g., [14]):

$$\begin{array}{lll}\min & \sum_{t=0}^{T} (c^I I_t + c^B B_t) & \\ \text{s.t.} & B_t - I_t = \sum_{j=0}^{t} (d_j - x_j), & t = 0,\dots,T,\\ & x_t = 0, & t \neq k \cdot P, k = 0,\dots,N,\\ & & t = 0,\dots,T,\\ & x_t, B_t, I_t \geq 0, & t = 0,\dots,T.\end{array} \qquad (2)$$

The problem (2) can solved in $O(T)$. For each $k = 0,\dots,N-1$, we determine periods:

$$\begin{aligned}h^k = \max\{t \ : t \in \{&k \cdot P,\dots,(k+1)\cdot P - 1\},\\ &c^I(t - k\cdot P) \leq c^B((k+1)\cdot P - t)\}.\end{aligned}$$

An optimal production plan to (2) is computed by the following formula for $t = 0,\dots,T$ and $k = 0,\dots,N$:

$$x_t = \begin{cases} 0 & \text{if } t \neq k \cdot P,\\ \mathbf{D}_{h^k} & \text{if } t = k\cdot P \text{ and } k = 0,\\ \mathbf{D}_{h^k} - \mathbf{D}_{h^{k-1}} & \text{if } t = k \cdot P \text{ and } 0 < k < N,\\ \mathbf{D}_{k\cdot P} - \mathbf{D}_{h^{k-1}} & \text{if } t = k\cdot P \text{ and } k = N.\end{cases}$$

We have assumed that an initial inventory $I$ and an initial backorder $B$ are equal to zero. Otherwise, one can easily modify the above method to cope with $I > 0$ or $B > 0$.

## III. Robust Version of the Problem

In Section II, we have assumed the all input parameters in problem (1) are precisely known. However, in real life this is rarely the case. Here, we admit uncertainty on the demands.

### A. Model of uncertainty

One of the simplest form of the uncertainty representations is modeling the imprecise demands $\tilde{d}_t$, $t = 0,\dots,T$, as closed intervals $[\underline{d}_t, \overline{d}_t]$, $\underline{d}_t \geq 0$, where $\underline{d}_t$ and $\overline{d}_t$ are a minimal and a maximal possible values of demand $\tilde{d}_t$ in period $t$, respectively. So, assigning some interval $[\underline{d}_t, \overline{d}_t]$ to demand $\tilde{d}_t$ means that it will take some value within the interval, but it is not possible to predict at present which one, i.e. $d_t \in [\underline{d}_t, \overline{d}_t]$. From the above model of uncertainty, it follows that the imprecision of cumulative demand $\mathbf{D}_t = \sum_{i=0}^{t} d_i$ increases in subsequent periods, i.e. $\mathbf{D}_t \in [\sum_{i=0}^{t} \underline{d}_t, \sum_{i=0}^{t} \overline{d}_t]$. In fact, practitioners often express the knowledge on demand uncertainty by the range. The demand can be interpreted in two different ways: a demand in the period or a cumulative demand. For instance, a practitioner expresses the uncertainty on the demand by range $\pm\Delta$. If this uncertainty is interpreted as the one on demands in periods, then it leads to the cumulative demands with increasing uncertainty (see Fig. 1a)). This case is unrealistic compared to the case when the uncertainty is interpreted as the one on cumulative demands (see Fig. 1b)). So in this paper, the uncertainty of the demands $\tilde{d}_t$ is described by the uncertainty on the cumulative demands, modeled by intervals $[\underline{\mathbf{D}}_t; \overline{\mathbf{D}}_t]$, instead of the uncertainty on demands in periods,

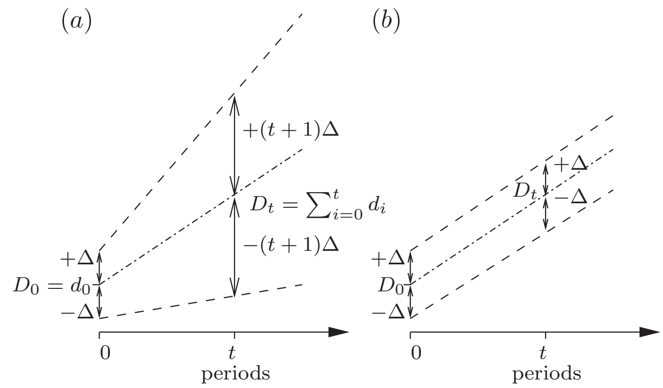


Fig. 1. An example: (a) the case with the uncertainty on demands in periods and the resulting uncertain cumulative demands, (b) the case with the uncertainty on cumulative demands.

modeled by $[\underline{d}_t, \overline{d}_t]$. Hence, we are given intervals $[\underline{\mathbf{D}}_t; \overline{\mathbf{D}}_t]$ that model the uncertainty of the cumulative demands for each period $t$, $t = 0, \ldots, T$, where $\underline{\mathbf{D}}_t$ and $\overline{\mathbf{D}}_t$ are a minimal and a maximal possible values of cumulative demands in period $t$, respectively. Obviously, $\underline{\mathbf{D}}_{t-1} \leq \underline{\mathbf{D}}_t$ and $\overline{\mathbf{D}}_{t-1} \leq \overline{\mathbf{D}}_t$, $t = 1, \ldots, T$.

A vector $S = (\mathbf{D}_0, \ldots, \mathbf{D}_T)$, $\mathbf{D}_t \in [\underline{\mathbf{D}}_t; \overline{\mathbf{D}}_t]$, $\mathbf{D}_{t-1} \leq \mathbf{D}_t$, that represents an assignment of cumulative demands $\mathbf{D}_t$ to periods $t$, $t = 0, \ldots, T$, is called a *scenario*. Thus every scenario expresses a realization of the cumulative demands. It is easy to check that scenario $S = (\mathbf{D}_0, \ldots, \mathbf{D}_T)$ induces an assignment of demands in periods $t$, $t = 0, \ldots, T$. Namely, $d_t = \mathbf{D}_t - \mathbf{D}_{t-1}$. We denote by $\Gamma$ the set of all the scenarios, i.e.

$$\Gamma = \{S = (\mathbf{D}_0, \ldots, \mathbf{D}_T) : \mathbf{D}_t \in [\underline{\mathbf{D}}_t; \overline{\mathbf{D}}_t], t = 0, \ldots, T, \\ \mathbf{D}_{t-1} \leq \mathbf{D}_t, t = 1, \ldots, T\}.$$

Among the scenarios of $\Gamma$, we distinguish the ones called *extreme scenarios*. Each extreme scenario $S = (\mathbf{D}_t)_{t=0}^T$ belongs to the set of scenarios defined by the following recurrence formula:

$$\mathbf{D}_t \in \begin{cases} \{\underline{\mathbf{D}}_0, \overline{\mathbf{D}}_0\} & \text{if } t = 0, \\ \{\max\{\mathbf{D}_{t-1}, \underline{\mathbf{D}}_t\}, \overline{\mathbf{D}}_t\} & \text{if } t = 1, \ldots, T. \end{cases} \tag{3}$$

We will denoted by $\Gamma_{\text{ext}}$ the set of extreme scenarios. Clearly, $\Gamma_{\text{ext}} \subseteq \Gamma$. The cumulative demand and the demand in period $t$ under scenario $S$ are denoted by $\mathbf{D}_t(S)$, $\mathbf{D}_t(S) \in [\underline{\mathbf{D}}_t; \overline{\mathbf{D}}_t]$, and $d_t(S)$, respectively, $d_t(S) = \mathbf{D}_t(S) - \mathbf{D}_{t-1}(S)$. Clearly, for every $S \in \Gamma$ it holds $\mathbf{D}_{t-1}(S) \leq \mathbf{D}_t(S)$, $t = 1, \ldots, T$. The function $C_t(\mathbf{X}_t, \mathbf{D}_t(S)) = \max\{c^I(\mathbf{X}_t - \mathbf{D}_t(S)), c^B(\mathbf{D}_t(S) - \mathbf{X}_t)\}$, represents either the cost of storing inventory from period $t$ to period $t+1$ or the cost of backordering quantity from period $t+1$ to period $t$ under scenario $S$. Now $F(\boldsymbol{x}, S)$ denotes the total cost of a production plan $\boldsymbol{x} \in \mathbb{X}$ under scenario $S$, i.e. $F(\boldsymbol{x}, S) = \sum_{t=0}^T C_t(\mathbf{X}_t, \mathbf{D}_t(S))$. The set feasible production amounts $\mathbb{X}$ is the same as in Section II.

In order to choose a robust production plan, one of robust criteria, called the *min-max* can be adopted (see, e.g. [10]). In the *min-max* version of problem (1), we seek a feasible production plan with the minimum the worst total cost over all scenarios, that is

$$\begin{aligned} \text{ROB}: \min_{\boldsymbol{x} \in \mathbb{X}} A(\boldsymbol{x}) &= \min_{\boldsymbol{x} \in \mathbb{X}} \max_{S \in \Gamma} F(\boldsymbol{x}, S) \\ &= \min_{\boldsymbol{x} \in \mathbb{X}} \max_{S \in \Gamma} \sum_{t=0}^T C_t(\mathbf{X}_t, \mathbf{D}_t(S)). \end{aligned}$$

In other words, we wish to find among all production plans the one that minimizes the maximum production plan cost over all scenarios, that minimizes $A(\boldsymbol{x})$, $A(\boldsymbol{x})$ is the *maximal cost of production plan* $\boldsymbol{x}$. An optimal solution $\boldsymbol{x}^r$ to the problem ROB is called *optimal robust production plan*.

Here and subsequently (as in Section II), we assume that an initial inventory $I$ and an initial backorder $B$ are equal to zero. Otherwise, one can modify the algorithms presented in this section to cope with the case $I > 0$ or $B > 0$.

### *B. Evaluating a Given Production Plan*

In this section, we will be concerned with evaluating a given production plan. We will propose methods for computing the optimal interval containing all possible values of costs of the production plan.

Let $\boldsymbol{x}^* \in \mathbb{X}$ be a given production plan. A scenario $S^o \in \Gamma$ that minimizes the total cost $F(\boldsymbol{x}^*, S)$ of the production plan $\boldsymbol{x}^*$ is called *optimistic scenario*. A scenario $S^w \in \Gamma$ that maximizes the total cost $F(\boldsymbol{x}^*, S)$ of the production plan $\boldsymbol{x}^*$ is called *the worst case scenario*. Thus, the optimal interval containing all possible values of costs of the production plan $\boldsymbol{x}^*$ is of form: $[F(\boldsymbol{x}^*, S^o), F(\boldsymbol{x}^*, S^w)]$. We begin with a result on function $F(\boldsymbol{x}^*, S) = \sum_{t=0}^T C_t(\mathbf{X}_t, \mathbf{D}_t(S))$:

*Proposition 1:* Function $F(\boldsymbol{x}^*, S)$ is convex on $\Gamma$ for any fixed production plan $\boldsymbol{x}^* \in \mathbb{X}$.

Proposition 1 follows by similar arguments as in [7], i.e. function $c^I(\mathbf{X}_t^* - \mathbf{D}_t(S))$ and $c^B(\mathbf{D}_t(S) - \mathbf{X}_t^*)$ are convex on $\Gamma$ and so $\max\{c^I(\mathbf{X}_t^* - \mathbf{D}_t(S)), c^B(\mathbf{D}_t(S) - \mathbf{X}_t^*)\}$ and $\sum_{t=0}^T \max\{c^I(\mathbf{X}_t^* - \mathbf{D}_t(S)), c^B(\mathbf{D}_t(S) - \mathbf{X}_t^*)\}$ are convex.

*1) Computing an Optimistic Scenario:* The problem of determining an optimistic scenario $S^o = (\mathbf{D}_t^o)_{t=0}^T$ for a given production plan $\boldsymbol{x}^* \in \mathbb{X}$, i.e. the problem

$$F(\boldsymbol{x}^*, S^o) = \min_{S \in \Gamma} F(\boldsymbol{x}^*, S), \tag{4}$$

can be formulated by a linear programming problem:

$$\begin{array}{lll} \min & \sum_{t=0}^T (c^I I_t + c^B B_t) & \\ \text{s.t.} & B_t - I_t = \mathbf{D}_t - \mathbf{X}_t^*, & t = 0, \ldots, T, \\ & \mathbf{D}_{t-1} \leq \mathbf{D}_t, & t = 1, \ldots, T, \\ & \underline{\mathbf{D}}_t \leq \mathbf{D}_t \leq \overline{\mathbf{D}}_t, & t = 0, \ldots, T. \\ & B_t, I_t \geq 0, & t = 0, \ldots, T \end{array} \tag{5}$$

If $\mathbf{D}_t^o$, $B_t^o$ and $I_t^o$ is an optimal solution to (5), then $S^o = (\mathbf{D}_t^o)_{t=0}^T$ is an optimistic scenario for $\boldsymbol{x}^*$. Furthermore, since $c^I, c^b \geq 0$, either $I_t^o > 0$ or $B_t^o > 0$, which means that for $S^o$ storing from period $t$ to $t+1$ and backordering from period $t+1$ to $t$ are not performed simultaneously. However, determining an optimistic scenario for $\boldsymbol{x}^*$ can be improved to $O(T)$, since one can give the explicit form of an optimistic scenario $S^o = (\mathbf{D}_t^o)_{t=0}^T$ and thus an optimal solution to (5):

$$\mathbf{D}_t^o = \begin{cases} \underline{\mathbf{D}}_t & \text{if } \mathbf{X}_t^* < \underline{\mathbf{D}}_t, \\ \mathbf{X}_t^* & \text{if } \underline{\mathbf{D}}_t \le \mathbf{X}_t^* \le \overline{\mathbf{D}}_t,\ t = 0,\dots,T. \\ \overline{\mathbf{D}}_t & \text{if } \mathbf{X}_t^* > \overline{\mathbf{D}}_t, \end{cases} \tag{6}$$

The form (6) follows from inequalities: $\underline{\mathbf{D}}_{t-1} \le \underline{\mathbf{D}}_t$, $\overline{\mathbf{D}}_{t-1} \le \overline{\mathbf{D}}_t$, $\mathbf{X}_{t-1}^* \le \mathbf{X}_t^*$, $t = 1,\dots,T$, and Proposition 1.

*2) Computing a Worst Case Scenario:* We now pass on to the problem of computing a worst case scenario $S^w = (\mathbf{D}_t^w)_{t=0}^T$ for a given production plan $\boldsymbol{x}^* \in \mathbb{X}$, i.e.

$$F(\boldsymbol{x}^*, S^w) = \max_{S \in \Gamma} F(\boldsymbol{x}^*, S). \tag{7}$$

Problem (7) is more difficult than the one of computing an optimistic scenario and thus a solution algorithm is much more involved. The following proposition allows us to construct an efficient algorithm for the problem under consideration:

*Proposition 2:* A worst case scenario $S^w$ is an extreme one, i.e. $S^w \in \Gamma_{\text{ext}}$.

*Proof:* Proposition follows by the same method as [7]. Function $F(\boldsymbol{x}^*, S)$ attains its maximum in convex set $\Gamma$. An easy computation shows that scenarios $S \in \Gamma_{\text{ext}}$ are the vertices of $\Gamma$. From the above and Proposition 1, it follows that $F(\boldsymbol{x}^*, S)$ attains the maximum value at a vertex of $\Gamma$ (see, e.g., [15]). ■

We now construct an algorithm for the problem (7), based on a *dynamic programming technique*. Let $\mathbb{D}_t$ be the set of feasible cumulative demand levels in period $t$, $t = 0,\dots,T$. Namely $\mathbf{D}_t \in \mathbb{D}_t$ if and only if $\mathbf{D}_t$ is the $t$th component of an extreme scenario that belongs to the set generated by formula (3) ($\Gamma_{\text{ext}}$). Let $\mathscr{C}_{t-1}(\mathbf{D}_{t-1})$ be the maximal cost of a given production plan $\boldsymbol{x}^*$ over periods $t,\dots,T$, when the cumulative demand level up to period $t-1$ is equal to $\mathbf{D}_{t-1}$, $\mathbf{D}_{t-1} \in \mathbb{D}_{t-1}$, $\mathscr{C}_{t-1} : \mathbb{D}_{t-1} \to \mathbb{R}_{\ge 0}$. Set $\mathbb{D}_{-1} = \{0\}$. We see at once that:

$$\mathscr{C}_T(\mathbf{D}_T) = 0, \mathbf{D}_T \in \mathbb{D}_T, \tag{8}$$

$$\mathscr{C}_{t-1}(\mathbf{D}_{t-1}) = \max \left\{ \begin{array}{l} C_t(\mathbf{X}_t^*, \overline{\mathbf{D}}_t) + \mathscr{C}_t(\overline{\mathbf{D}}_t) \\ C_t(\mathbf{X}_t^*, \max\{\mathbf{D}_{t-1}, \underline{\mathbf{D}}_t\}) + \\ + \mathscr{C}_t(\max\{\mathbf{D}_{t-1}, \underline{\mathbf{D}}_t\}) \end{array} \right\}$$
$$\mathbf{D}_{t-1} \in \mathbb{D}_{t-1}, t = T,\dots,0. \tag{9}$$

The maximal cost of production plan $\boldsymbol{x}^*$ over period $0,\dots,T$ is equal to $\mathscr{C}_{-1}(0)$, $\mathscr{C}_{-1}(0) = F(\boldsymbol{x}^*, S^w)$, which is computed by the backward recursion (8) and (9). Worst case scenario $S^w = (\mathbf{D}_t^w)_{t=0}^T$ for $\boldsymbol{x}^*$ can be determined by a forward recursion technique. It is sufficient to store for each $\mathbf{D}_{t-1} \in \mathbb{D}_{t-1}$ the value for which the maximum in (9) is attained, that is $\mathbf{D}_t^w$ is either $\max\{\mathbf{D}_{t-1}, \underline{\mathbf{D}}_t\}$ or $\underline{\mathbf{D}}_t$.

Let us analyze the running time of the dynamic programming based algorithm. Building the sets of feasible cumulative demand levels $\mathbb{D}_0,\dots,\mathbb{D}_T$ according to formula (3) and computing $F(\boldsymbol{x}^*, S^w)$ by the backward recursion (8) and (9) can be done in $O(T \cdot \max_{t=0,\dots,T} |\mathbb{D}_t|)$. Determining $S^w$ by by a forward recursion takes $O(T)$. It is easy to check that $\max_{t=0,\dots,T} |\mathbb{D}_t| \le T + 2$. Hence, the total running time of the algorithm is $O(T^2)$.

### C. Computing an Optimal Robust Production Plan

In this section, we will propose algorithms for computing an optimal robust production plan to problem ROB. We will first construct an algorithm for problem ROB without any additional restrictions on ROB. We will then put some restrictions on ROB and obtain more efficient algorithms for computing an optimal robust production plan.

*1) Algorithm for the General Problem* ROB*:* We now give an iterative algorithm for the problem ROB based on iterative adding constraints for min-max problems proposed in [16]. Similar methods were developed for min-max regret linear programming problems with an interval objective function [17], [18] and a production planning problem with interval demands [7]. Let us perform a relaxation of the problem ROB that consists in replacing a given cumulative demand scenario set $\Gamma$ with a discrete scenario set $\Gamma_{\text{dis}} = \{S^1,\dots,S^K\}$, $\Gamma_{\text{dis}} \subseteq \Gamma$:

$$\begin{array}{rll} \text{RX-ROB:} & \hat{z} = \min z & \\ \text{s.t.} & z \ge F(\boldsymbol{x}, S^i) & \forall S^i \in \Gamma_{\text{dis}}, \\ & \boldsymbol{x} \in \mathbb{X}, & \end{array} \tag{10}$$

where $S^i = (\mathbf{D}_t^i)_{t=0}^T, \boldsymbol{x} = (x_t)_{t=0}^T$. Since $\Gamma_{\text{dis}} \subseteq \Gamma$, $\hat{z}$ is a lower bound on the maximal cost of an optimal robust production plan $\boldsymbol{x}^r$ to problem ROB, $\hat{z} \le A(\boldsymbol{x}^r) = F(\boldsymbol{x}^r, S^w)$. Note that the constraint $v \ge F(\boldsymbol{x}, S^i)$ called *scenario cut*, associated with $S^i$ is not a linear constraint. Each scenario cut associated with $S^i$ can be linearized by replacing it with the following $T{+}2$ constraints and $2T{+}2$ new decision variables ($B_t^i$ and $I_t^i$):

$$\begin{array}{ll} z \ge \sum_{t=0}^{T} (c^I I_t^i + c^B B_t^i), & \\ B_t^i - I_t^i = \mathbf{D}_t^i - \sum_{j=0}^{t} x_j, & t = 0,\dots,T, \\ B_t^i, I_t^i \ge 0, & t = 0,\dots,T. \end{array} \tag{11}$$

Replacing each scenario cut $z \ge F(\boldsymbol{x}, S^i)$ by (11) in (10) leads to the following linear program:

$$\begin{array}{lll} & \hat{z} = \min z & \\ \text{s.t.} & z \ge \sum_{t=0}^{T} (c^I I_t^i + c^B B_t^i), & \forall S^i \in \Gamma_{\text{dis}}, \\ & B_t^i - I_t^i = \mathbf{D}_t^i - \sum_{j=0}^{t} x_j, & t = 0,\dots,T, \forall S^i \in \Gamma_{\text{dis}}, \\ & B_t^i, I_t^i \ge 0, & t = 0,\dots,T, \forall S^i \in \Gamma_{\text{dis}}, \\ & x_t = 0, & t \ne k \cdot P,\ k = 0,\dots,N, \\ & & t = 0,\dots,T, \\ & x_t \ge 0, & t = 0,\dots,T. \end{array} \tag{12}$$

The iterative algorithm for the problem ROB (Algorithm 1) starts with zero lower bound $LB = 0$, a candidate $\hat{\boldsymbol{x}} \in \mathbb{X}$ for an optimal solution for ROB (any solution in $\mathbb{X}$) and empty discrete scenario set, $\Gamma_{\text{dis}} = \emptyset$. At each iteration, a worst case scenario $S^w$ for $\hat{\boldsymbol{x}}$ is determined by the dynamic programming algorithm presented in Section III-B2. The value of $A(\hat{\boldsymbol{x}}) = F(\hat{\boldsymbol{x}}, S^w)$ is an upper bound on $A(\boldsymbol{x}^r)$, $A(\boldsymbol{x}^r) \leq A(\hat{\boldsymbol{x}})$. If a termination criterion is fulfilled (Step 3), for a given precision $\epsilon > 0$, then the algorithm stops with production plan $\hat{\boldsymbol{x}}$ being an approximation of an optimal robust production plan $\boldsymbol{x}^r$. Otherwise the worst case scenario $S^w$ is added to $\Gamma_{\text{dis}}$, the scenario cut corresponding to $S^w$ is appended to problem RX-ROB or equivalently to linear programming problem (12) . Next the updated linear programming problem (12) is solved to obtain a better candidate $\hat{\boldsymbol{x}}$ for an optimal robust production plan $\boldsymbol{x}^r$ to problem ROB and new lower bound $LB = \hat{z}$. Since set $\Gamma_{\text{dis}}$ is updated during the course of the algorithm, the computed values of lower bounds $\{\hat{z}\}$ form a nondecreasing sequence of their values. Then new iteration is started.

**Algorithm 1:** Solving problem ROB.

**Input**: $[\underline{\mathbf{D}}_t, \overline{\mathbf{D}}_t,]$, $t = 0, \dots T$, $c^I$, $c^B$, initial production plan $\hat{\boldsymbol{x}} \in \mathbb{X}$, a tolerance $\epsilon > 0$.
**Output**: A production plan, an approximation of an optimal robust production plan, and its worst case scenario.
**Step** 0. $i := 0$, $LB := 0$, $\Gamma_{\text{dis}} := \emptyset$.
**Step** 1. $\boldsymbol{x}^i := \hat{\boldsymbol{x}}$.
**Step** 2. Compute a worst case scenario $S^w$ for $\boldsymbol{x}^i$ by the dynamic programming algorithm presented in Section III-B2.
**Step** 3. $\Delta := F(\boldsymbol{x}^i, S^w) - LB$. If $LB > 1$ then $\Delta := \Delta / LB$. If $\Delta \leq \epsilon$ then output $\boldsymbol{x}^i$, $S^w$ and STOP.
**Step** 4. $i := i + 1$, $S^i := S^w$, $\Gamma_{\text{dis}} := \Gamma_{\text{dis}} \cup \{S^i\}$ and append scenario cut $z \geq F(\boldsymbol{x}, S^i)$ to problem RX-ROB.
**Step** 5. Compute an optimal solution $(\hat{\boldsymbol{x}}, \hat{z})$ to RX-ROB (linear programming problem (12)), $LB := \hat{z}$, and go to Step 1.

Algorithm 1 terminates in a finite number of iterations for any given $\epsilon > 0$. In order to show this, the same reasoning applies as those given in [19, Theorem 2.5], [16, Theorem 3] and [7, Theorem 3]. It is worth pointing out that the running time of each iteration highly depends on Step 2 and Step 5, where a worst case scenario is computed and linear programming problem (12) is solved. The running time of Step 2 is $O(T^2)$, linear program (12) (Step 5) can be solved by using a specially-tuned method for this kind of problems [20] or by some standard off-the-shelf LP solvers. Hence, each iteration of Algorithm 1 can be done efficiently (in a polynomial time).

*2) Algorithms for Special Cases of* ROB*:* Consider the problem ROB, when periodicity $P = 1$. In this case, we can apply a method proposed in [7] for computing an optimal robust production plan $\boldsymbol{x}^r = (x_t^r)_{t=0}^T$:

$$\mathbf{X}_t^r = \frac{c^B \overline{\mathbf{D}}_t + c^I \underline{\mathbf{D}}_t}{c^B + c^I}, \quad t = 0, \dots, T. \tag{13}$$

Set $x_0^r = \mathbf{X}_0^r$ and $x_t^r = \mathbf{X}_t^r - \mathbf{X}_{t-1}^r$ for $t = 1, \dots, T$.

Consider the problem ROB, when $P > 1$ (if $P = 1$ one can use (13)) and the bounds of cumulative demand intervals are such that $\overline{\mathbf{D}}_{t-1} \leq \underline{\mathbf{D}}_t$ for every $t = 1, \dots, T$. According to the above assumption, the set of cumulative demand scenarios $\Gamma$ has the form $\Gamma = [\underline{\mathbf{D}}_0, \overline{\mathbf{D}}_0] \times \cdots \times [\underline{\mathbf{D}}_T, \overline{\mathbf{D}}_T]$. Now each $S = (\mathbf{D}_t)_{t=0}^T \in \Gamma$ has components such that $\mathbf{D}_{t-1} \leq \mathbf{D}_t$, $t = 1, \dots, T$. Hence, and the periodicity in the problem ROB, it follows that one can decompose the problem into $N + 1$ separate subproblems, that is:

$$\begin{aligned}
\boldsymbol{x}^r &= \arg \min_{\boldsymbol{x} \in \mathbb{X}} \max_{S \in \Gamma} \sum_{t=0}^{T} C_t(\mathbf{X}_t, \mathbf{D}_t(S)) = \\
&= \sum_{k=0}^{N-1} \min_{\mathbf{X}_{k \cdot P} \in \mathcal{X}^k} \max_{S \in \Gamma^k} \sum_{t=k \cdot P}^{(k+1) \cdot P - 1} C_t(\mathbf{X}_{k \cdot P}, \mathbf{D}_t(S)) \\
&+ \min_{\mathbf{X}_{N \cdot P} \in \mathcal{X}^N} \max_{S \in \Gamma^N} C_{N \cdot P}(\mathbf{X}_{N \cdot P}, \mathbf{D}_{N \cdot P}(S)),
\end{aligned} \tag{14}$$

where $\mathcal{X}^k = [\underline{\mathbf{D}}_{k \cdot P}, \overline{\mathbf{D}}_{k \cdot P}] \cup \cdots \cup [\underline{\mathbf{D}}_{(k+1) \cdot P - 1}, \overline{\mathbf{D}}_{(k+1) \cdot P - 1}]$, $\Gamma^k = [\underline{\mathbf{D}}_{k \cdot P}, \overline{\mathbf{D}}_{k \cdot P}] \times \cdots \times [\underline{\mathbf{D}}_{(k+1) \cdot P - 1}, \overline{\mathbf{D}}_{(k+1) \cdot P - 1}]$, $k = 0, \dots, N-1$, $\mathcal{X}^N = [\underline{\mathbf{D}}_{N \cdot P}, \overline{\mathbf{D}}_{N \cdot P}]$ and $\Gamma^N = [\underline{\mathbf{D}}_{N \cdot P}, \overline{\mathbf{D}}_{N \cdot P}]$; $\Gamma = \Gamma^0 \times \cdots \times \Gamma^N$. Obviously, the above possible cumulative production levels are nondecreasing sequence of their values, $\mathbf{X}_0 \leq \mathbf{X}_P \leq \cdots \leq \mathbf{X}_{N \cdot P}$. Therefore, we need only to solve the $N + 1$ separate subproblems. The last subproblem is trivial, i.e. an optimal cumulative production level $\mathbf{X}_{N \cdot P}^r$ can be computed by formula:

$$\mathbf{X}_{N \cdot P}^r = \frac{c^B \overline{\mathbf{D}}_{N \cdot P} + c^I \underline{\mathbf{D}}_{N \cdot P}}{c^B + c^I}. \tag{15}$$

It remains to solve the subproblems for $k = 0, \dots, N - 1$, i.e.

$$\min_{\mathbf{X}_{k \cdot P} \in \mathcal{X}^k} \max_{S \in \Gamma^k} \sum_{t=k \cdot P}^{(k+1) \cdot P - 1} C_t(\mathbf{X}_{k \cdot P}, \mathbf{D}_t(S)). \tag{16}$$

Consider the $k$th subproblem. Since $\mathbf{X}_t = \mathbf{X}_{k \cdot P}$ in period $t$, $t \in \{k \cdot P + 1, \dots, (k + 1) \cdot P - 1\}$, worst case scenarios $S^w \in \Gamma^k$ for $\mathbf{X}_{k \cdot P}$ belong to the set $[\underline{\mathbf{D}}_{k \cdot P}, \underline{\mathbf{D}}_{k \cdot P}] \times \cdots \times [\underline{\mathbf{D}}_{h-1}, \underline{\mathbf{D}}_{h-1}] \times [\underline{\mathbf{D}}_h, \overline{\mathbf{D}}_h] \times [\overline{\mathbf{D}}_{h+1}, \overline{\mathbf{D}}_{h+1}] \times \cdots \times [\overline{\mathbf{D}}_{(k+1) \cdot P - 1}, \overline{\mathbf{D}}_{(k+1) \cdot P - 1}] \subseteq \Gamma^k$, assuming that $\mathbf{X}_{k \cdot P} \in [\underline{\mathbf{D}}_h, \overline{\mathbf{D}}_h]$.

For each $h = k \cdot P, \dots, (k + 1) \cdot P - 1$, we compute a possible cumulative production level

$$\mathbf{X}_{k \cdot P}^h = \frac{c^B \overline{\mathbf{D}}_h + c^I \underline{\mathbf{D}}_h}{c^B + c^I} \in [\underline{\mathbf{D}}_h, \overline{\mathbf{D}}_h].$$

Note that for $\mathbf{X}_{k \cdot P}^h$ equality $C_h(\mathbf{X}_{k \cdot P}^h, \underline{\mathbf{D}}_h) = C_h(\mathbf{X}_{k \cdot P}^h, \overline{\mathbf{D}}_h)$ holds and its worst case scenario has form $(\underline{\mathbf{D}}_{k \cdot P}, \dots, \underline{\mathbf{D}}_{h-1}, \overline{\mathbf{D}}_h, \dots, \overline{\mathbf{D}}_{(k+1) \cdot P - 1})$. Thus, the worst case cost for $\mathbf{X}_{k \cdot P}^h$ over the set of scenarios $\Gamma^k$, denoted by $C^h$, is as follows:

$$\begin{aligned}
C^h(\mathbf{X}_{k \cdot P}^h) &= \max_{S \in \Gamma^k} \sum_{t=k \cdot P}^{(k+1) \cdot P - 1} C_t(\mathbf{X}_{k \cdot P}^h, \mathbf{D}_t(S)) \\
&= \sum_{t=k \cdot P}^{h-1} C_t(\mathbf{X}_{k \cdot P}^h, \underline{\mathbf{D}}_t) + \sum_{t=h}^{(k+1) \cdot P - 1} C_t(\mathbf{X}_{k \cdot P}^h, \overline{\mathbf{D}}_t).
\end{aligned}$$

We then determine an optimal cumulative production level $\mathbf{X}_{k \cdot P}^r$ for the $k$th subproblem (16) by formula:

$$\mathbf{X}_{k \cdot P}^r = \arg \min_{h \in \{k \cdot P, \dots, (k+1) \cdot P - 1\}} C^h(\mathbf{X}_{k \cdot P}^h). \tag{17}$$

Using (15) and (17), one can easily compute an optimal robust production plan $\boldsymbol{x}^r$. Thus, (14) can be solved in $O(T)$.

## IV. Conclusion

In this paper, we have studied the possibility of integrating the uncertainty into planning process. We have proposed efficient algorithms for solving the production planning problem using MRP process with the POQ rule under uncertain cumulative demands, modeled by closed intervals, with the minmax criterion. Namely, the algorithm based on iterative adding constraints. It is worth pointing out that each iteration can done in a polynomial time. Moreover, we have provided algorithms for two special cases of the problem under consideration. The first case corresponds to the Lot-For-Lot (L4L) rule. In the second one the range of uncertainty is small compared to the demand.

An interesting topic for further research is investigating the production planning problem using MRP process under uncertain cumulative demands, considered in this paper, with other rules such as: Fixed Order Quantity (FOQ) and Minimal Order Quantity (MOQ), and with production constraints.

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